

Fractional Maps and Fractional Attractors.

Part II: Fractional Difference Caputo α -Families of Maps

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Abstract: In this paper we extend the notion of an α -family of maps to discrete systems defined by simple difference equations with the fractional Caputo difference operator. The equations considered are equivalent to maps with falling factorial-law memory which is asymptotically power-law memory. We introduce the fractional difference Universal, Standard, and Logistic α -Families of Maps and propose to use them to study general properties of discrete nonlinear systems with asymptotically power-law memory.

Keywords:

fractal derivative, fractional difference, attractor, discrete map, power-law, memory

1. INTRODUCTION

Systems with memory are common in biology, social sciences, physics, and engineering (see review Edelman (2014c)). The most frequently encountered type of memory in natural and engineering systems is power-law memory. This leads to the possibility of describing them by fractional differential equations, which have power-law kernels. Nonlinear integro-differential fractional equations are difficult to simulate numerically - this is why in Tarasov and Zaslavsky (2008) the authors introduced fractional maps, which are equivalent to fractional differential equations of nonlinear systems experiencing periodic delta function-kicks, and proposed to use them for the investigation of general properties of nonlinear fractional dynamical systems. Bifurcation diagrams in the fractional Logistic Map related to a scheme of numerical integration of fractional differential equations were considered by Stanislavsky (2006).

An adequate description of discrete natural systems with memory can be obtained by using fractional difference equations (see Miller and Ross (1989); Gray and Zhang (1988); Agarwal (2000); Atici and Eloe (2009); Anastassiou (2009); Chen et al. (2011); Wu et al. (2014); Wu and Baleanu (2014)). In Chen et al. (2011); Wu et al. (2014); Wu and Baleanu (2014) the authors demonstrated that in some cases fractional difference equations are equivalent to maps (which we will call fractional difference maps) with falling factorial-law memory, where falling factorial function is defined as

$$t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}. \quad (1)$$

Falling factorial-law memory is asymptotically power-law memory -

$$\lim_{t \rightarrow \infty} \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)t^\alpha} = 1, \quad \alpha \in \mathbb{R}, \quad (2)$$

and we may expect that fractional difference maps have properties similar to the properties of fractional maps.

The goal of the present paper is to introduce fractional difference families of maps depending on memory and non-linearity parameters consistent with the previous research of fractional maps (see Edelman (2014c); Tarasov and Zaslavsky (2008); Edelman and Tarasov (2009); Tarasov (2009a,b, 2011); Edelman (2011); Edelman and Taieb (2013); Edelman (2013a,b)) in order to prepare a background for an investigation of general properties of systems with asymptotically power-law memory.

In the next section (Sec. 2) we will remind the reader how the regular Universal, Standard, and Logistic Maps (see Chirikov (1979.); Lichtenberg and Lieberman (1992); Zaslavsky (2008); May (1976)) are generalized to obtain fractional Caputo α -Families of Maps (α FM). In Sec. 3 we'll present some basics on fractional difference/sum operators, which will be used in Sec. 4 to derive the fractional difference Caputo Universal, Standard, and Logistic α FMs. In Sec. 5 we'll present some results on properties of fractional difference Caputo Standard α FM.

2. FRACTIONAL α -FAMILIES OF MAPS

Fractional α FM were introduced in Edelman (2013a), further investigated in Edelman (2013b), and reviewed in Edelman (2014c). The Universal α FM was obtained by integrating the following equation:

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$$\frac{d^\alpha x}{dt^\alpha} + G_K(x(t - \Delta)) \sum_{k=-\infty}^{\infty} \delta(t - (k + \varepsilon)) = 0, \quad (3)$$

where $\varepsilon > \Delta > 0$, $\alpha \in \mathbb{R}$, $\alpha > 0$, $\varepsilon \rightarrow 0$, with the initial conditions corresponding to the type of fractional derivative to be used. $G_K(x)$ is a nonlinear function which depends on the nonlinearity parameter K . It is called Universal because integration of Eq. (3) in the case $\alpha = 2$ and $G_K(x) = KG(x)$ produces the regular Universal Map (see Zaslavsky (2008)). In what follows the author considers Eq. (3) with the left-sided Caputo fractional derivative (see Samko et al. (1993); Kilbas et al. (2006); Podlubny (1999))

$$\begin{aligned} {}^C_0 D_t^\alpha x(t) &= I_t^{N-\alpha} D_t^N x(t) \\ &= \frac{1}{\Gamma(N-\alpha)} \int_0^t \frac{D_\tau^N x(\tau) d\tau}{(t-\tau)^{\alpha-N+1}}, \quad (N = [\alpha]), \end{aligned} \quad (4)$$

where $N \in \mathbb{Z}$, $D_t^N = d^N/dt^N$, ${}_0 I_t^\alpha$ is a fractional integral, $\Gamma()$ is the gamma function, and the initial conditions are

$$(D_t^k x)(0+) = b_k, \quad k = 0, \dots, N-1. \quad (5)$$

There are two reasons to restrict the consideration in this paper to the Caputo case (the Riemann-Liouville case won't be considered): a) as in the case of fractional differential equations, in the case of fractional difference equations it is much easier to define initial conditions for Caputo difference equations than for Riemann-Liouville difference equations; b) the main goal of this work is to compare fractional and fractional difference maps, and the case of Caputo maps serves the purpose. Comparison of the Riemann-Liouville and Caputo Standard Maps was considered in Edelman (2011).

The problem Eqs. (3)–(5) is equivalent to the Volterra integral equation of the second kind ($t > 0$) Kilbas et al. (2006)

$$\begin{aligned} x(t) &= \sum_{k=0}^{N-1} \frac{b_k}{k!} t^k \\ &- \frac{1}{\Gamma(\alpha)} \int_0^t d\tau \frac{G_K(x(\tau - \Delta))}{(t-\tau)^{1-\alpha}} \sum_{k=-\infty}^{\infty} \delta(\tau - (k + \varepsilon)). \end{aligned} \quad (6)$$

After the introduction $x^{(s)}(t) = D_t^s x(t)$ the Caputo Universal α FM can be written as (see Tarasov (2011))

$$\begin{aligned} x_{n+1}^{(s)} &= \sum_{k=0}^{N-s-1} \frac{x_0^{(k+s)}}{k!} (n+1)^k \\ &- \frac{1}{\Gamma(\alpha-s)} \sum_{k=0}^n G_K(x_k) (n-k+1)^{\alpha-s-1}, \end{aligned} \quad (7)$$

where $s = 0, 1, \dots, N-1$ and $x_0^{(k+s)} = b_{k+s}$.

In the case $G_K(x) = K \sin(x)$ and $\alpha = 2$ with $p = x^{(1)}$ Eq. (7) produces the well-known Standard Map (see Chirikov (1979)), which on a torus can be written as

$$p_{n+1} = p_n - K \sin(x_n), \quad (\text{mod } 2\pi), \quad (8)$$

$$x_{n+1} = x_n + p_{n+1}, \quad (\text{mod } 2\pi). \quad (9)$$

This is why the Caputo Universal α FM Eq. (7) with

$$G_K(x) = K \sin(x) \quad (10)$$

is called the Caputo Standard α FM:

$$\begin{aligned} x_{n+1}^{(s)} &= \sum_{k=0}^{N-s-1} \frac{x_0^{(k+s)}}{k!} (n+1)^k \\ &- \frac{K}{\Gamma(\alpha-s)} \sum_{k=0}^n \sin(x_k) (n-k+1)^{\alpha-s-1}, \end{aligned} \quad (11)$$

where $s = 0, 1, \dots, N-1$.

In the case $G_K(x) = x - Kx(1-x)$ and $\alpha = 1$ Eq. (7) produces the well-known Logistic Map (see May (1976))

$$x_{n+1} = Kx_n(1-x_n). \quad (12)$$

This is why the Caputo Universal α FM Eq. (7) with

$$G_K(x) = G_{LK}(x) = x - Kx(1-x) \quad (13)$$

is called the Caputo Logistic α FM:

$$\begin{aligned} x_{n+1}^{(s)} &= \sum_{k=0}^{N-s-1} \frac{x_0^{(k+s)}}{k!} (n+1)^k \\ &- \frac{1}{\Gamma(\alpha-s)} \sum_{k=0}^n \frac{x - Kx(1-x)}{(n-k+1)^{1+s-\alpha}}, \end{aligned} \quad (14)$$

where $s = 0, 1, \dots, N-1$.

The Caputo Standard and Logistic α FMs were investigated in detail in Edelman (2014c, 2013a,b) for the case $\alpha \in (0, 2]$ which is important in applications.

- For $\alpha = 0$ the Caputo Standard and Logistic α FMs are identically zeros: $x_n = 0$.
- For $0 < \alpha < 1$ the Caputo Standard α FM is

$$x_n = x_0 - \frac{K}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \frac{\sin(x_k)}{(n-k)^{1-\alpha}}, \quad (\text{mod } 2\pi). \quad (15)$$

and the Caputo Logistic α FM is

$$x_n = x_0 - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \frac{x - Kx(1-x)}{(n-k)^{1-\alpha}}. \quad (16)$$

- For $\alpha = 1$ the 1D Standard Map is the Circle Map with zero driving phase

$$x_{n+1} = x_n - K \sin(x_n), \quad (\text{mod } 2\pi). \quad (17)$$

and the 1D Logistic α FM is the Logistic Map Eq. (12).

- For $1 < \alpha < 2$ the Caputo Standard α FM is

$$\begin{aligned} p_{n+1} &= p_n - \frac{K}{\Gamma(\alpha-1)} \left[\sum_{i=0}^{n-1} V_\alpha^2(n-i+1) \sin(x_i) \right. \\ &\quad \left. + \sin(x_n) \right], \quad (\text{mod } 2\pi), \end{aligned} \quad (18)$$

$$\begin{aligned} x_{n+1} &= x_n + p_0 - \frac{K}{\Gamma(\alpha)} \sum_{i=0}^n V_\alpha^1(n-i+1) \sin(x_i), \\ &(\text{mod } 2\pi), \end{aligned} \quad (19)$$

where $V_\alpha^k(m) = m^{\alpha-k} - (m-1)^{\alpha-k}$ and the Caputo Logistic α FM is

$$x_{n+1} = x_0 + p(n+1)^k - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^n [x_k - Kx_k(1-x_k)](n-k+1)^{\alpha-1}, \quad (20)$$

$$p_{n+1} = p_0 - \frac{1}{\Gamma(\alpha-1)} \sum_{k=0}^n [x_k - Kx_k(1-x_k)](n-k+1)^{\alpha-2}. \quad (21)$$

- For $\alpha = 2$ the Caputo Standard Map is the regular Standard Map as in Eqs. (8) and (9) above. The 2D Logistic Map is

$$p_{n+1} = p_n + Kx_n(1-x_n) - x_n, \quad (22)$$

$$x_{n+1} = x_n + p_{n+1}. \quad (23)$$

3. FRACTIONAL DIFFERENCE/SUM OPERATORS

In this paper we will adopt the definition of the fractional sum ($\alpha > 0$)/difference ($\alpha < 0$) operator introduced in Miller and Ross (1989) as

$${}_a\Delta_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s). \quad (24)$$

Here f is defined on \mathbb{N}_a and ${}_a\Delta_t^{-\alpha}$ on $\mathbb{N}_{a+\alpha}$, where $\mathbb{N}_t = \{t, t+1, t+2, \dots\}$, and falling factorial $t^{(\alpha)}$ is defined by Eq. (1). As Miller and Ross noticed, their way to introduce the discrete fractional sum operator based on the Green's function approach is not the only way to do so. In Gray and Zhang (1988) the authors defined the discrete fractional sum operator generalizing the n -fold summation formula in a way similar to the way in which the fractional Riemann–Liouville integral is defined in fractional calculus by extending the Cauchy n -fold integral formula to the real variables. They mentioned the following theorem but didn't present a proof.

Theorem 1. For $\forall n \in \mathbb{N}$

$$\begin{aligned} {}_a\Delta_t^{-n} f(t) &= \frac{1}{(n-1)!} \sum_{s=a}^{t-n} (t-s-1)^{(n-1)} f(s) \\ &= \sum_{s^0=a}^{t-n} \sum_{s^1=a}^{s^0} \dots \sum_{s^{n-1}=a}^{s^{n-2}} f(s^{n-1}), \end{aligned} \quad (25)$$

where s^i , $i = 0, 1, \dots, n-1$ are the summation variables.

Proof. Indeed, this formula is obviously true for $n = 1$. Let's assume that Eq. (25) is true for $n-1$:

$$\begin{aligned} {}_a\Delta_t^{-(n-1)} f(t) &= \sum_{s=a}^{t-(n-1)} C(t-s-1, n-2) f(s) \\ &= \sum_{s^1=a}^{t-(n-1)} \sum_{s^2=a}^{s^1} \dots \sum_{s^{n-1}=a}^{s^{n-2}} f(s^{n-1}), \end{aligned} \quad (26)$$

where $C(i, j)$ is the number of j -combinations from a given set of i elements. Then, for $t = s^0 + n - 1$ Eq. (26) gives

$$\sum_{s^1=a}^{s^0} C(s^0 - s^1 + n - 2, n - 2) f(s^1)$$

$$= \sum_{s^1=a}^{s^0} \sum_{s^2=a}^{s^1} \dots \sum_{s^{n-1}=a}^{s^{n-2}} f(s^{n-1}). \quad (27)$$

Now Eq. (25) can be obtained from

$$\begin{aligned} &\sum_{s^0=a}^{t-n} \sum_{s^1=a}^{s^0} \dots \sum_{s^{n-1}=a}^{s^{n-2}} f(s^{n-1}) \\ &= \sum_{s^0=a}^{t-n} \sum_{s^1=a}^{s^0} C(s^0 - s^1 + n - 2, n - 2) f(s^1) \\ &= \sum_{s^1=a}^{t-n} f(s^1) \sum_{s^0=s^1}^{t-n} C(s^0 - s^1 + n - 2, n - 2) \\ &= \sum_{s^1=a}^{t-n} C(t - s^1 - 1, n - 1) f(s^1) = {}_a\Delta_t^{-n} f(t). \end{aligned} \quad (28)$$

Here we used the identity

$$\sum_{s^0=s^1}^{t-n} C(s^0 - s^1 + n - 2, n - 2) = C(t - s^1 - 1, n - 1), \quad (29)$$

which is true for $t = n + s^1$ and can be proven by induction for any t

$$\begin{aligned} &\sum_{s^0=s^1}^{t+1-n} C(s^0 - s^1 + n - 2, n - 2) = C(t - s^1 - 1, n - 2) \\ &+ C(t - s^1 - 1, n - 1) = C(t - s^1, n - 1). \end{aligned} \quad (30)$$

This ends the proof.

As we see, two different approaches are consistent with the definition of the fractional sum operator given by Miller and Ross (see also Atici and Eloe (2009)). For $\alpha > 0$ and $m - 1 < \alpha < m$ Anastassiou (2009) defined the fractional (left) Caputo-like difference operator as

$$\begin{aligned} {}_a^C\Delta_t^\alpha x(t) &= {}_a\Delta_t^{-(m-\alpha)} \Delta^m x(t) \\ &= \frac{1}{\Gamma(m-\alpha)} \sum_{s=a}^{t-(m-\alpha)} (t-s-1)^{(m-\alpha-1)} \Delta^m x(s), \end{aligned} \quad (31)$$

where Δ^m is the m -th power of the forward difference operator defined as $\Delta x(t) = x(t+1) - x(t)$. The proof (see Miller and Ross (1989) p.146) that ${}_0\Delta_t^\lambda$ in the limit $\lambda \rightarrow 0$ approaches the identity operator can be easily extended to the ${}_a\Delta_t^\lambda$ operator. In this case the definition Eq. (31) can be extended to all real $\alpha \geq 0$ with ${}_a^C\Delta_t^m x(t) = \Delta^m x(t)$ for $m \in \mathbb{N}_0$. Then, the Anastassiou's fractional Taylor difference formula Anastassiou (2009)

$$\begin{aligned} x(t) &= \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k x(a) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=a+m-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} {}_a^C\Delta_t^\alpha x(t), \end{aligned} \quad (32)$$

where x is defined on \mathbb{N}_a , $m = \lceil \alpha \rceil$, and $a \in \mathbb{N}_0$ for $\forall t \in \mathbb{N}_{a+m}$ is valid for any real $\alpha > 0$ and for integer $\alpha = m$ is identical to the integer discrete Taylor's formula (see p.28 in Agarwal (2000))

$$x(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k x(a) + \frac{1}{(m-1)!} \sum_{s=a}^{t-m} (t-s-1)^{(m-1)} \Delta^m x(t). \quad (33)$$

As it was noticed in Wu et al. (2014) and Wu and Baleanu (2014), Lemma 2.4 from Chen et al. (2011) on the equivalency of the fractional Caputo-like difference and sum equations can be extended to all real $\alpha > 0$ and formulated as follows:

Theorem 2. The Caputo-like difference equation

$${}_a^C \Delta_t^\alpha x(t) = f(t + \alpha - 1, x(t + \alpha - 1)) \quad (34)$$

with the initial conditions

$$\Delta^k x(a) = c_k, \quad k = 0, 1, \dots, m-1, \quad m = \lceil \alpha \rceil \quad (35)$$

is equivalent to the fractional sum equation

$$x(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k x(a) + \frac{1}{\Gamma(\alpha)} \times \sum_{s=a+m-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s + \alpha - 1, x(s + \alpha - 1)), \quad (36)$$

where $t \in \mathbb{N}_{a+m}$.

Here we should notice that the authors of Wu et al. (2014) and Wu and Baleanu (2014) didn't consider the Caputo difference operator with integer α . As a result, Theorem 2 is not valid for integer values of α with their definition $m = \lceil \alpha \rceil + 1$.

This theorem in the limiting sense can be extended to all real $\alpha \geq 0$. Indeed, taking into account that $\lim_{\alpha \rightarrow 0} {}_a^C \Delta_t^\alpha x(t) = x(t)$, Eq. (34) for $\alpha = 0$ turns into

$$x(t) = f(t-1, x(t-1)). \quad (37)$$

For $\alpha = 0$ the first sum on the right in Eq. (36) disappears and in the second sum the only remaining term with $s = t - \alpha$ in the limit $\alpha \rightarrow 0$ turns into $f(t-1, x(t-1))$.

4. FRACTIONAL DIFFERENCE α -FAMILIES OF MAPS

In the following we assume that f is a nonlinear function $f(t, x(t)) = -G_K(x(t))$ with the nonlinearity parameter K and adopt the Miller and Ross proposition to let $a = 0$. Now, with $x_n = x(n)$, Theorem 2 can be formulated as

Theorem 3. For $\alpha \in \mathbb{R}$, $\alpha \geq 0$ the Caputo-like difference equation

$${}_0^C \Delta_t^\alpha x(t) = -G_K(x(t + \alpha - 1)), \quad (38)$$

where $t \in \mathbb{N}_m$, with the initial conditions

$$\Delta^k x(0) = c_k, \quad k = 0, 1, \dots, m-1, \quad m = \lceil \alpha \rceil \quad (39)$$

is equivalent to the map with falling factorial-law memory

$$x_{n+1} = \sum_{k=0}^{m-1} \frac{\Delta^k x(0)}{k!} (n+1)^{(k)}$$

$$- \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{n+1-m} (n-s-m+\alpha)^{(\alpha-1)} G_K(x_{s+m-1}), \quad (40)$$

where $x_k = x(k)$ which we will call the fractional difference Caputo Universal α -Family of Maps.

The fractional difference Caputo Universal α FM is similar to the general form of the Caputo Universal α FM Eq. (7). Both of them can be written as

$$x_n = x_0 + \sum_{k=1}^{m-1} \frac{p^k(0)}{k!} n^{(k)} - \frac{1}{\Gamma(\alpha)} \sum_{k=M}^{n-1} W_\alpha(n-k) G_K(x_k), \quad (41)$$

where $p^k(0)$ are the initial value of momenta defined as $p^s = D_t^s x(t)$ for fractional maps and as $p^s(t) = \Delta^s x(t)$ for fractional difference maps; $M = 0$ for fractional maps and $M = m-1$ for fractional difference maps; $n^{(s)} = n^s$ for fractional maps and $n^{(s)} = \Gamma(n+1)/\Gamma(n+1-s)$ for the fractional difference maps. $W_\alpha(s) = s^{\alpha-1}$ for fractional maps and $W_\alpha(s) = \Gamma(s+\alpha-1)/\Gamma(s)$ for fractional difference maps. Asymptotically, both expressions for $W_\alpha(s)$ coincide because of Eq. (2).

4.1 Fractional Difference Universal α FM

Let's consider the case $\alpha = 2$. Then the difference Eq. (34) produces

$$\Delta^2 x_n = -G_K(x_{n+1}) \quad (42)$$

and the equivalent sum equation is

$$x_{n+1} = x_0 + \Delta x_0(n+1) - \sum_{s=0}^{n-1} (n-s) G_K(x_{s+1}). \quad (43)$$

After the introduction $p_n = \Delta x_{n-1}$ with the assumption $G_K(x) = KG(x)$ the map equations indeed can be written as the well-known 2D Universal Map

$$p_{n+1} = p_n - KG(x_n), \quad (44)$$

$$x_{n+1} = x_n + p_{n+1}, \quad (45)$$

which for $G(x) = \sin(x)$ produces the Standard Map Eqs. (8) and (9). In the rest of this paper we'll call Eq. (40) with $G_K(x) = K \sin(x)$ the fractional difference Caputo Standard α -Family of Maps.

In the case $\alpha = 1$ the fractional difference Caputo Universal α FM is

$$x_{n+1} = x_n - G_K(x_n), \quad (46)$$

which produces the Logistic Map if $G_K(x) = x - Kx(1-x)$. In the rest of this paper we'll call Eq. (40) with $G_K(x) = x - Kx(1-x)$ the fractional difference Caputo Logistic α -Family of Maps.

4.2 $\alpha = 0$ Difference Caputo Standard and Logistic α FM

- In the case $\alpha = 0$ the 0D Standard Map turns into the Sine Map (see, e.g., Lalescu (2010))

$$x_{n+1} = -K \sin(x_n), \quad (\text{mod } 2\pi). \quad (47)$$

- The 0D Logistic Map is

$$x_{n+1} = -x_n + Kx_n(1-x_n). \quad (48)$$

4.3 $0 < \alpha < 1$ Fractional Difference Caputo Standard and Logistic α FM's

- For $0 < \alpha < 1$ the fractional difference Standard Map is

$$x_{n+1} = x_0 - \frac{K}{\Gamma(\alpha)} \sum_{s=0}^n \frac{\Gamma(n-s+\alpha)}{\Gamma(n-s+1)} \sin(x_s), \quad (\text{mod } 2\pi), \quad (49)$$

which after the π -shift of the independent variable $x \rightarrow x + \pi$ coincides with the “fractional sine map” proposed in Wu et al. (2014).

- The fractional difference Logistic Map can be written as

$$x_{n+1} = x_0 - \frac{1}{\Gamma(\alpha)} \sum_{s=0}^n \frac{\Gamma(n-s+\alpha)}{\Gamma(n-s+1)} [x_s - Kx_s(1-x_s)]. \quad (50)$$

The fractional Logistic Map introduced in Wu and Baleanu (2014) does not converge to the Logistic map in the case $\alpha = 1$.

4.4 $\alpha = 1$ Difference Caputo Standard and Logistic α FM's

- The $\alpha = 1$ difference Caputo Standard α FM is identical to the Circle Map with zero driven phase Eq. (17). The map considered in Wu et al. (2014)

$$x_{n+1} = x_n + K \sin(x_n), \quad (\text{mod } 2\pi) \quad (51)$$

is obtained from this map by the substitution $x \rightarrow x + \pi$.

- The $\alpha = 1$ Difference Caputo Logistic α FM is the regular Logistic Map.

4.5 $1 < \alpha < 2$ Fractional Difference Caputo Standard and Logistic α FM's

- For $1 < \alpha < 2$ the fractional difference Standard Map is

$$x_{n+1} = x_0 + \Delta x_0(n+1) - \frac{K}{\Gamma(\alpha)} \times \sum_{s=0}^{n-1} \frac{\Gamma(n-s+\alpha-1)}{\Gamma(n-s)} \sin(x_{s+1}), \quad (\text{mod } 2\pi). \quad (52)$$

- The $1 < \alpha < 2$ fractional difference Logistic Map is

$$x_{n+1} = x_0 + \Delta x_0(n+1) - \frac{1}{\Gamma(\alpha)} \times \sum_{s=0}^{n-1} \frac{\Gamma(n-s+\alpha-1)}{\Gamma(n-s)} [x_{s+1} - Kx_{s+1}(1-x_{s+1})]. \quad (53)$$

Let's introduce $p_n = \Delta x_{n-1}$; then these maps can be written as 2D maps with memory:

- The fractional difference Standard Map is

$$p_n = p_1 - \frac{K}{\Gamma(\alpha-1)} \quad (54)$$

$$\times \sum_{s=2}^n \frac{\Gamma(n-s+\alpha-1)}{\Gamma(n-s+1)} \sin(x_{s-1}), \quad (\text{mod } 2\pi),$$

$$x_n = x_{n-1} + p_n, \quad (\text{mod } 2\pi), \quad n \geq 1, \quad (55)$$

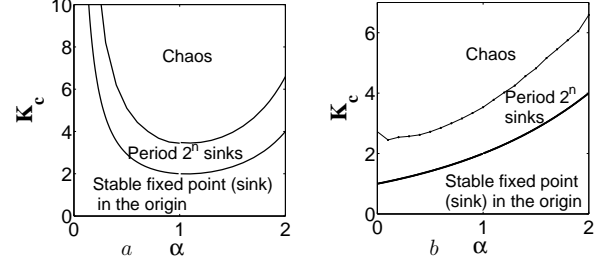


Fig. 1. $\alpha - K$ (bifurcation) diagrams for the Caputo (a) and Fractional Difference Caputo (b) Standard α FM's.

which in the case $x_0 = 0$ is identical to the “fractional standard map” introduced in Wu et al. (2014) (Eq. (18) with $\nu = \alpha - 1$ there).

- The fractional difference Logistic Map is

$$p_n = p_1 - \frac{K}{\Gamma(\alpha-1)} \quad (56)$$

$$\times \sum_{s=2}^n \frac{\Gamma(n-s+\alpha-1)}{\Gamma(n-s+1)} [x_{s-1} - Kx_{s-1}(1-x_{s-1})],$$

$$x_n = x_{n-1} + p_n, \quad n \geq 1. \quad (57)$$

4.6 $\alpha = 2$ Difference Caputo Standard and Logistic α FM's

- The $\alpha = 2$ difference Caputo Standard α FM is the regular Standard Map Eqs. (8) and (9).
- From Eqs. (44) and (45) the 2D difference Caputo Logistic α FM is

$$p_{n+1} = p_n - x_n + Kx_n(1-x_n), \quad (58)$$

$$x_{n+1} = x_n + p_{n+1}, \quad (59)$$

which is identical to the 2D Logistic Map Eqs. (22) and (23).

5. CONCLUSION

As we saw in Sec. 3, the fractional difference operator is a natural extension of the difference operator. The simplest fractional difference equations (of the Eq. (34) type), where the fractional difference on the left side is equal to a nonlinear function on the right side, are equivalent to maps with falling factorial-law (asymptotically power-law) memory Eq. (36). Systems with power-law memory play an important role in nature (see Edelman (2014c)) and investigation of their general properties is important for understanding behavior of natural systems.

Properties of the fractional difference Caputo Standard α FM were investigated in detail in Edelman (2014a) (see also Sec. 3 in Edelman (2014b)). Qualitatively, properties of the fractional difference and fractional maps (maps with falling factorial- and power-law memory) are similar. The similarity reveals itself in the dependence of systems' properties on the memory (α) and nonlinearity (K) parameters (bifurcation diagrams, see Figs. 1, 2, and 3), power-law convergence to attractors, non-uniqueness of solutions (intersection of trajectories and overlapping of attractors), and cascade of bifurcations and intermittent cascade of bifurcations type behaviors (see Figs. 4 and 5).

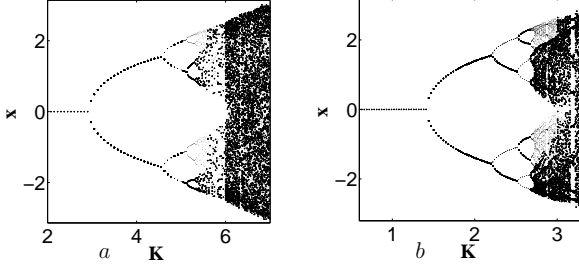


Fig. 2. Bifurcation diagrams for the Caputo (a) and Fractional Difference Caputo (b) Standard α FMs with $\alpha = 0.5$ obtained after 5000 iterations with the initial condition $x_0 = 0.1$ (regular points) and $x_0 = -0.1$ (bold points).

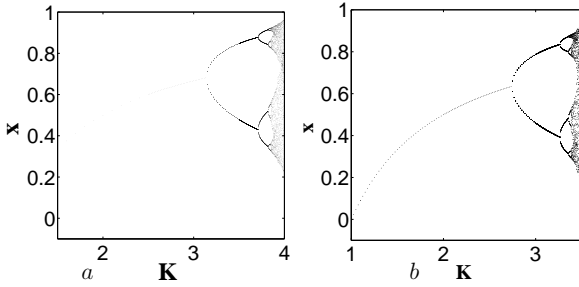


Fig. 3. Bifurcation diagrams for the Caputo (a) and Fractional Difference Caputo (b) Logistic α FMs with $\alpha = 0.8$ obtained after 1000 iterations with the initial condition $x_0 = 0.01$.

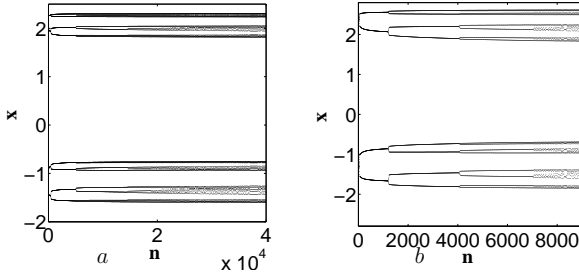


Fig. 4. Cascade of bifurcations type trajectories for the Caputo (a) and Fractional Difference Caputo (b) Standard α FMs with $\alpha = 0.1$ and $x_0 = 0.1$. $K = 26.65$ in (a) and $K = 2.41$ in (b).

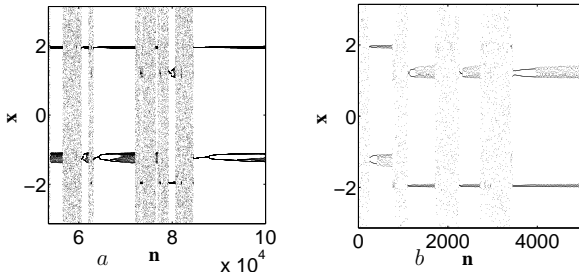


Fig. 5. Intermittent cascade of bifurcations type trajectories for the Caputo (a) and Fractional Difference Caputo (b) Standard α FMs. In (a) $\alpha = 1.65$, $K = 4.5$, $x_0 = 0.$, and $p_0 = 0.3$. In (b) $\alpha = 1.5$, $K = 4.82$, $x_0 = 0.$, and $p_0 = 0.01$.

The differences of the properties of the falling factorial-law memory maps from the power-law memory maps are

the results of the differences in weights of the recent (with $(n - j)/n \ll 1$) values of the maps' variables at the time instants t_j in the definition of the present values at time t_n and are significant when $\alpha \in (0, 1)$ (especially when $\alpha \rightarrow +0$), see Fig. 1.

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